On quality properties of eigenvalues of Euler-Bernoulli beams under axial loads

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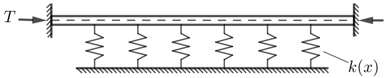
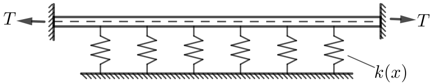
Content

Formulation of the problem

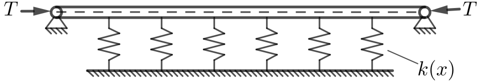
We study the eigenvalues of a uniform Euler-Bernoulli beam with axial loads, lying on a Winkler-type foundation with two types of fastening at the ends:

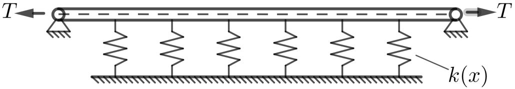
- clamped-clamped. $k(x)$ is the variable coefficient of foundation.





- Hinged-hinged,





Let $k(x)$, $x \in (0;1)$ be a real-valued summable function, T is the axial load, λ are the eigenvalues. The foundation reaction force is:

$$p(x,t) = k(x)w(x,t),$$

where $w(x,t)$ is the transverse displacement.
Natural frequencies of the Euler-Bernoulli beam for the linear $k(x) = k_0(1 - \alpha x)$, $0 \leq \alpha \leq 1$ and the nonlinear function $k(x) = k_0(1 - \beta x^2)$, $0 \leq \beta \leq 1$, were investigated in papers [1, 2].
In [3] the explicit form of the solution of the fourth order differential equation for $k(x) = \frac{1}{(c_0+c_1x)^4}$ with hinged-hinged of fixing and at constant external load was obtained.

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[1] Zhou, D.; A general solution to vibrations of beams on variable winkler elastic foundation. Computers and Structures, 47(1), (1993) 83–90

[2] Kacar, A., Tan, H.T., Kaya, M.O.; Free vibration analysis of beams on variable Winkler elastic foundation by using the differential transformation method. Mathematical and Computational Applications, Vol. 16, No. 3, pp. 773-783, 2011

[3] Froio, D., Rizzi, E.; Analytical solution for the elastic bending of beams lying on a variable Winkler support // Acta Mech 227, (2016) 1157-1179

- The boundary value problems for ordinary high-order differential operators in which the spectrum is absent or the spectrum is a countable set were considered in the case with constant coefficients in [4] and with variable coefficients in [5].
- Recently, based on the numerical methods for estimation of the eigenvalues: the Haar wavelet method [6], the polynomial expansion and the integral technique methods [7], the Chebyshev spectral Method [8]

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[4] Locker, J.; Eigenvalues and completeness for regular and simply irregular two-point differential operators, Mem. of the AMS, 195 (2008), 1-177.

[5] Makin, A.; Two-point boundary-value problems with nonclassical asymptotics on the spectrum. Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 95, pp. 1–7.

[6] Z. Shi, Y. Cao; Application of Haar wavelet method to eigenvalue problems of high order differential equations, Appl. Math. Model. 36 (2012) 4020-4026.

[7] Huang, Y., Chen, J., Luo, Qi-Zhi; A simple approach for determining the eigenvalues of the fourth-order Sturm-Liouville problem with variable coefficients. Appl. Math. Letters 26 (2013) 729-734.

[8] Agarwal, P., Attary, M., Maghasedi, M., Kumam, P.; Solving Higher-Order Boundary and Initial Value Problems via Chebyshev–Spectral Method: Application in Elastic Foundation, Symmetry 2020, 12, 987

- The spectrum asymptotics of the boundary value problems for ordinary fourth-order differential operators with periodic and antiperiodic conditions were obtained in [9] and with matrix coefficients with various special cases in [10].
- Asymptotics of eigenvalues for the Euler–Bernoulli operator with various coefficients (real, complex and its converge to the constant function) were studied in [11].

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[9] Polyakov D.M. Spectral analysis of a fourth order differential operator with periodic and antiperiodic boundary conditions. St. Petersburg Mathematical Journal, 27:5 (2016), 789–811

[10] Polyakov D. M. Spectral estimates for the fourth-order operator with matrix coefficients. Comp. Math. Math. Phys., 60:7 (2020), 1163–1184

[11] Badanin A., Korotyaev E. Inverse problems and sharp eigenvalue asymptotics for Euler–Bernoulli operators. Inverse Problems, (31) 2015 055004 (37pp)

Clamped fastening at points $x = 0, x = 1$

The problem of transverse vibrations of a beam of unit length

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + k(x)w(x,t) + EJ\frac{\partial^4 w(x,t)}{\partial x^4} + T\frac{\partial^2 w(x,t)}{\partial x^2} = 0.$$

after replacement $w(x,t) = v(\lambda, x)sin(\omega t)$ reduces to the following spectral problem:

$$EJv^{IV}(\lambda, x) + Tv''(\lambda, x) + k(x)v(\lambda, x) = \lambda v(\lambda, x), \quad x \in I_p, p = 1, 2, \tag{1}$$

$$v(\lambda, 0) = 0,\ v'(\lambda, 0) = 0, v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0, \tag{2}$$

where $I_1 = (0,\ 1),\ I_2 = \left(\frac{1}{2},\ 1\right)$. We also introduce the following boundary conditions

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0 \tag{3}$$

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0 \tag{4}$$

- In [12] a closed form of the natural frequencies of various boundary value problems for the equation (1) with $k(x) = 0$ was obtained and the known results from [13, 14] were modified. The symmetric equivalence of boundary value problems was also investigated.
- The eigenvalues of a hinged beam at the end points on a constant elastic foundation can be found in [15, P. 437], [16, P. 106]
- The eigenvalues of a hinged beam at the end points on a constant elastic foundation with an axial load can be found in [17, P. 148].

[12] Valle J., Fernandez D., Madrenas J. Closed-form equation for natural frequencies of beams under full range of axial loads modeled with a spring-mass system. International Journal of Mechanical Sciences, **153-154** (2019), 380–390.

[13] Bokaian A. Natural frequencies of beams under compressive axial loads. Journal of Sound and Vibration, **126**:1 (1988), 49–65.

[14] Bokaian A. Natural frequencies of beams under tensile axial loads. Journal of Sound and Vibration, **142**:3 (1990), 481–498.

[15] L. Meirovitch, Analytical methods in vibrations. Toronto, Ontario, 1967.

[16] Robert D. Blevins, Formulas for natural frequency and mode shape. Litton Educational Publishing, Inc., 1979.

[17] Lawrence N. Virgin, Vibration of Axially Loaded Structures. Cambridge University Press, New York, 2007.

- The influence of a constant foundation coefficient on the critical load was investigated in [[18]].
- With a variable foundation coefficient without axial load, the symmetric equivalence of boundary value problems for a uniform beam was investigated in [[19]].

[18] Shvartsman B., Majak J. Numerical method for stability analysis of functionally graded beams on elastic foundation. Applied Mathematical Modelling, **40** (2016), 3713–3719.

[19] Nurakhmetov D., Jumabayev S., Aniyarov A., Kussainov R. Symmetric properties of eigenvalues and eigenfunctions of uniform beams. *Symmetry*, **12** 2097 (2020), 1–13

Main result

Let $\sigma(A_1), \sigma(B_1), \sigma(C_1)$ be a set of eigenvalues of problems $A_1 - \lambda I$, $B_1 - \lambda I$, $C_1 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (2), (3), (4), respectively.

Theorem (1)

Let $k(x)$ be a symmetric function with respect to the point $x = \frac{1}{2}$

$$k(x) = k(1-x), \quad x \in [0;1]$$

and $T < T_{cr}$. The following statements are true:
1. $\sigma(A_1) \equiv \sigma(B_1) \cup \sigma(C_1)$
2. If $\lambda \in \sigma(B_1)$ or $\lambda \in \sigma(C_1)$, then the eigenfunctions of problems $A_1 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

Hinged fastening at points $x = 0, x = 1$

$$v(\lambda, 0) = 0,\ v''(\lambda, 0) = 0,\ v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0 \tag{5}$$

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0, \tag{6}$$

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0. \tag{7}$$

Let $\sigma(A_2), \sigma(B_2), \sigma(C_2)$ be a set of eigenvalues of problems $A_2 - \lambda I$, $B_2 - \lambda I$, $C_2 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (5), (6), (7), respectively.

Theorem (2)

Let $k(x)$ be a symmetric function with respect to the point $x = \frac{1}{2}$ and $T < T_{cr}$. The following statements are true:
1. $\sigma(A_2) \equiv \sigma(B_2) \cup \sigma(C_2)$
2. If $\lambda \in \sigma(B_2)$ or $\lambda \in \sigma(C_2)$, then the eigenfunctions of problems $A_2 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

Example

Example (1)

Let $k(x) = 4x(1-x)$, $EJ = 1$ and $T = 10$. In this example $T_{cr} = 39.55$.

Table: Numerical calculations of the first four eigenvalues from the example 1

Clamped-clamped at the points $x = 0$, $x = 1$	Sliding at the poin $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
377.66	377.66	3342.79
3342.79	13628.94	38228.41
13628.94	86495.98	170138.82
38228.41	303256.54	512382.96

Example (2)

Let $k(x) = 100(x^7 + 1)$, $EJ = 1$ and $T = 20$. In this example $T_{cr} = 47.19$. The function $k(x)$ does not satisfy the symmetry condition.

Table: Numerical calculations of the first four eigenvalues from the example 2.

Clamped-clamped at the points $x = 0$, $x = 1$	Sliding at the poin $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
353.83	356.3	2990.49
2984.42	12754.87	36630.55
12746.56	83975.72	166498.41
36620.91	298298.52	505917.18

Example (3)

Let $k(x) = 4x(1-x)$, $EJ = 1$ and $T = -10$. The function $k(x)$ satisfies the symmetry condition. Comparative analysis: the known behavior of natural frequencies is preserved with a variable coefficient $k(x)$ [a].

[a] Valle J., Fernandez D., Madrenas J. Closed-form equation for natural frequencies of beams under full range of axial loads modeled with a spring-mass system. International Journal of Mechanical Sciences, **153-154** (2019), 380–390.

Table: Numerical calculations of the first four eigenvalues from the example 3.

Clamped-clamped at the points $x = 0$, $x = 1$	Sliding at the poin $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
623.78	623.78	4263.84
4263.84	15606.99	41660.1
15606.99	91775.92	177661.44
41660.1	313415.93	525563.87

Example (4)

Let $k(x) = 4x(1-x)$, $EJ = 1$ and $T = 5$. In this example $T_{cr} = 9.96$.

Table: Numerical calculations of the first four eigenvalues from the example 4.

Hinged-hinged at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, hinged at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, hinged at the point $x = 1$
48.93	48.93	1361.87
1361.87	7446.69	24147.84
7446.69	59647.65	124466.33
24147.84	231461.85	395830.03

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Thank you for attention.